Science and Technology - Chaos and Fractals

1 – THE SCIENCE AND TECHNOLOGY SERIES

In order to promote and support the development of the scientific and technological knowledge in the S.A.R. of Macao - objectives delineated in the Government's Policy Outlining - the Macao Post has been publishing for the past few years philatelic issues such as: The Composition and Construction of DNA, The Standard Model of Particle Physics, and The Cosmology XXI, which not only conform to the aforementioned theme, but may also be considered part of a group of fundamental questions still unanswered that have tormented the human intellect for thousands of years.

In this 4th Series, called **Chaos** and **Fractals**, we aim once again to offer the many collectors of Macao Philately, all over the world, a theme that is contemporary, modern, and at the frontiers of science.

The following text does not opt for the formalism that characterizes Mathematics, rather tries, through everyday language, though as rigorous as possible, to transmit some of the basic and most popular facts on this subject.

If, after reading it, your curiosity is picked enough to desire to broaden your own knowledge about **Chaos Theory** and **Fractal Geometry**, then this issue will have fulfilled one of its goals.

2 – CHAOS THEORY AND FRACTAL GEOMETRY

Chaos and **Fractals**, over the last decades, have captured the attention and interest of economists, biologists, physicists, etc., as well as of the public in general, due to how they have been able to create a new understanding of the complexity in nature. In fact, Chaos Theory and Fractal Geometry managed to change and correct a very limited vision and conception of the world around us.

The Chaos Theory describes complex motion and the dynamics of sensitive systems. Chaos addresses the issue of whether or not it is possible to make precise long-term predictions of any system if the initial conditions are known to an accurate degree. The Chaos Theory has a variety of applications, namely to predict patterns in weather, stock markets, human mental states, etc.

Fractal Geometry is a kind of "language" used to describe, model and analyse complex forms, non-uniform shapes and rough edges found in Nature. While the elements of the traditional Euclidean Geometry are visual, such as lines, circles, cubes, etc., those in Fractal Geometry are invisible, as the algorithms - a set of instructions on how to create them. The first studies about Fractal Geometry were the works of **Gaston Julia**, at the beginning of the 20th Century. Some of its applications fall in the fields of Seismology, Cosmology, Image Compression, etc.

3 – WHAT IS A FRACTAL?

Benoît Mandelbrot, the Polish-born mathematician who created the term Fractal, said, "I coined the word Fractal in 1975 from the Latin ***fractus*** which describes a broken stone - fragmented and irregular".

Structures that are presently called Fractals, like the **Cantor Set**, the **Peano Curve** or the **Koch Snowflake**, were not only known and explored before Mandelbrot coined them, but were also considered, at that time, as mathematical monsters, aberrations or pathological objects.

To create a Fractal we take a simple shape and transform it successively according to a set of rules, in an iterative or recursive process.

Fractals are very complex geometric shapes with:

- Structural **Self-similarity** - look the same at different magnifications;
- **Indefinite Resolution** - details are infinite; and
- **Fractional Dimension**.

4 – FRACTAL DIMENSION

To explain the concept of **Fractal Dimension** it is necessary to understand what Dimension means. In Euclidean Geometry: a **point**, has no dimension - no length, no width, and no height; a **line segment**, has one dimension - length; an **area**, has two dimensions - length and width; a **volume**, has three dimensions - length, width and height.

Why is a line segment one-dimensional and an area two-dimensional?

We can break a **line segment**, which is one-dimensional, into 2, 3, 4...or ***k*** **Self-similar** pieces. Taking one of them, to yield the original, we should amplify it by a **Magnification Factor** of 2, 3, 4...or ***k***, respectively.

A square is not the same. We can break a **square**, which is bi-dimensional, into 4, 9, 16...k^2 Self-similar sub-squares. Taking one of them, to yield the original, we should amplify it by a Magnification Factor of 2, 3, 4... ***k***, respectively. As a general rule, a square can be broken into k^2 Self-similar copies of itself, each of which must be magnified by a factor ***k*** to yield the original square.

As a general rule for the **cube**, which is tri-dimensional, we can break it into k^3 Self-similar pieces, each of which must be magnified by a factor ***k*** to yield the original cube.

From the above examples we can find a way to specify the **Dimension** of a **Self-similar object**. That is, considering a Self-similar structure, there is a relationship between the **Magnification Factor** k ($k = 1/r$, where r is the **Reduction Factor**) and the number of Self-similar pieces n given by the power law:

$$n = k^d = 1/r^d$$ where d is the **Dimension**.

Applying logarithms:

Fractal Dimension $$d = \frac{\log n}{\log k}$$

n = number of Self-similar pieces
k = Magnification Factor

5 – THE HILBERT CURVE

The **Hilbert Curve** was introduced by the German mathematician **David Hilbert** (1862-1943). Among several achievements, Hilbert is considered the father of Axiomatic Geometry, and was the proponent, in 1900, of 23 outstanding mathematical problems whose solution he thought might be found by the mathematicians during the 20th Century.

The line, the plane, and the space, are perceived as having one, two and three dimensions, respectively. The Hilbert Curve is a curve with one dimension that fills a plane or a space, i.e., which passes through every point within a given plane or space.

To construct the Hilbert Curve: [See Stamp: Hilbert Curve, S071 (6/1)]

- Take a dotted square you are going to fill with a curve;
- Divide it into four smaller squares and connect their centres with three line segments to form the Curve C_0 with the shape of an inverted "U";
- Next, create four copies of C_0 reduced by ½ and place them in the four smaller squares after rotating the bottom-left one 90° clockwise and the bottom-right one 90° anticlockwise;
- Then, connect the start and end points of the four curves with three line segments half the size previously used, to obtain the Curve C_1;
- Next, reduce C_1 by ½ and place, once again, four copies in the four smaller squares;
- As above, repeat the process, rotate the bottom squares 90° clockwise (the left one) and 90° anticlockwise (the right one);
- Connect the four C_1 curves with three line segments reduced to ¼ of the original size, to obtain the Curve C_2;
- C_2 contains 16 copies of C_0, each ¼ of its size.

This process may be repeated as many times as you wish.

6 – THE FRACTAL TREE

According to **Benoît Mandelbrot** and **Michael Frame**, the construction of a **Binary Fractal Tree** [See Stamp: Fractal Tree, S071 (6/2)] is defined recursively by symmetric binary branching: "The **trunk** of length l splits into **two branches** of length rl where r is the **Reduction Factor**, each describing an angle θ with the direction of the trunk. Each next branch then splits by the same rule".

In other words, each of the two initial branches splits into two branches of length r^2l, each describing an angle θ with the direction of its parent branch.

Continuing this process as many times as we wish, the Tree is the set of branches together with their terminal points, called **branch tips**.

Each branch is defined by a sequence of symbols L (Left) and R (Right) according to the direction taken along the Tree to reach the corresponding branch tip.

A Fractal Tree is defined by three parameters:

- A length l (of the trunk);
- An angle θ (between the trunk and the first branch);
- A ratio r (of the successive branches' lengths).

7 – THE SIERPINSKI TRIANGLE

The **Sierpinski Triangle** or **Gasket** was introduced by the Polish mathematician **Waclaw Sierpinski** (1882-1969). Sierpinski, Kuratowski, Banach and others, belong to the so-called "Polish School" and worked together in the emerging field of "Abstract Spaces".

The Sierpinski Triangle is constructed starting with a red triangle in a plane to which we apply the following repetitive process: [See Stamp: Sierpinski Triangle S071 (6/3)]

Take the three midpoints of the sides. These points with the vertices of the initial triangle define four small triangles of which we remove the central one. Apply the same procedure to the three remaining triangles and repeat as many times as you wish.

The Sierpinski Triangle (Gasket) is the set of points in the plane that we obtain when the above-mentioned process is repeated *ad infinitum*. The red area of a Sierpinski Triangle is zero.

Because the Sierpinski Triangle is made of three copies of itself reduced by a factor of ½ its **Fractal Dimension** is **log3/log2≈1.585**.

8 – THE CHAOS GAME

The term **Chaos Game** was established by **Michael Barnsley** and its outcome is one of the most interesting Fractals, considering the process that creates it.

The Chaos Game is played as follows: [See Stamp: Chaos Game, S071 (6/4)]

- Get a pencil, a dice (with faces numbering 1, 1, 2, 2, 3 and 3), and a sheet of paper;
- On the paper mark three points numbering 1, 2 and 3, which constitute the vertices of a triangle (rectangle, equilateral, isosceles, or any other), and another arbitrarily chosen point Z_0, on the inside or outside of the triangle, called starting point or the "seed";
- Throw the dice; assume the result is two; move the seed to the midpoint between Z_0 and the vertice 2, call it Z_1;
- Throw the dice again; assume the result is one; move the seed to the midpoint between Z_1 and the vertice 1, call it Z_2;
- Repeat the previous process as often as you wish.

A pattern similar to the Sierpinski Triangle gradually emerges, which is amazing considering that the Sierpinski Triangle is a structure representing order and predictability. Surprisingly, we can see that **a random process can create a shape that is extremely organized and deterministic** - precisely the opposite of a random shape, what we would eventually expect.

Is God playing or is it pure coincidence?

9 – THE KOCH CURVE

The **Koch Curve** was introduced by the Swedish mathematician **Helge von Koch** (1870-1924). If we join together three properly rotated Koch Curves, a new figure is formed, which, for its similarities, is called the Snowflake Curve or the Koch Island.

The Koch Curve [See Stamp: Von Koch Curve, S071 (6/5)] is constructed by dividing a straight line - the **Initiator** - into three equal parts and replacing the middle one by an equilateral triangle without its base. This figure, composed of four segments, is called the **Generator** and will be reused in the following steps.

Repeat the process for each of the four existing segments, dividing them into three equal parts, etc. It results in a continuous curve of infinite length, Self-similar, and nowhere differentiable.

The Koch Curve is a curve made of corners everywhere, i.e., it is impossible to trace a tangent to any of its points.

Because the Koch Curve is made of four one-third-sized Koch Curves, its **Fractal Dimension** is **log4/log3≈1.262**.

10 – THE CANTOR SET

The **Cantor Set** was introduced by the German mathematician **Georg Cantor** (1845-1918), who became renowned with his work in the field we now call Set Theory.

The basic Cantor Set [See Stamp: Cantor Set, S071 (6/6)] is an infinite set of points in the unit interval [0,1], obtained by repeatedly removing the middle thirds of line segments.

The Cantor Set is **uncountable** and **Self-similar**. Because it is equal to two copies of itself if each copy is magnified by a factor of 3, its **Fractal Dimension** is equal to **log2/log3≈0.631**.

11 – THE JULIA AND MANDELBROT SETS

Gaston Julia (1893-1978) was a French mathematician who became famous in 1918 with the publication of his masterpiece *"Mémoire sur l'itération des fonctions rationnelles"*. Julia is considered one of the forefathers of modern Dynamical Systems Theory.

His work had been forgotten until the 1970s, when Mandelbrot brought it back to light.

Benoît Mandelbrot, a Polish-born French-American mathematician (1924 -), is known as the creator of Fractal Geometry and is largely responsible for the current interest in it. He showed how Fractals can occur both in Mathematics and elsewhere in Nature.

Mandelbrot, who then already worked at IBM, was able to demonstrate, with the aid of computer graphics, that Julia's work is the source of some of the most beautiful Fractals.

The **Mandelbrot Set** [See Souvenir Sheet: Mandelbrot Set, B064 (1/1)] is a Fractal that is defined as the set of points C, in the complex plane, for which the following sequence: $Z_0=0$; $Z_{n+1}=Z_n^2+C$, does not tend to infinity.

The Mandelbrot Set was first defined in 1905 by Pierre Fatou, who proved that once a point moved to a distance greater than 2 from the origin, the orbit - the sequence values of interaction - would escape to infinity.

The Julia Set [See Souvenir Sheet: Julia Set, B064 (1/1)] is a Fractal that is defined as the set of points $Z_0=Z$, in the complex plane, for which the following iterative sequence: $Z_0=Z$; $Z_{n+1}=Z_n^2+C$, does not tend to infinity. Depending on the choice of $Z_0=Z$ and C, a large range of orbits patterns is possible to be obtained.

Julia Set is closely related to Mandelbrot Set. The Mandelbrot Set is an index to the Julia Sets. For any point on the complex plane a corresponding Julia Set can be obtained. Where a point lies in the Mandelbrot Set, the Julia Set is connected; otherwise the Julia Set is a Cantor Dust of unconnected points.

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